Instructor: Jorge R. Viramontes
MATHEMATICS 2303

The Real Number System.

Definition. A **Set** is a collection of elements listed within braces. The set $\{a, b, c, d, e\}$ consists of five elements, namely a, b, c, d, and e. A set that contains no elements is called an **empty set** (or **null set**). The symbols $\{\ \}$, or ϕ are used to represent the empty set.

Definition. The following number sets are important:

- Natural Numbers: $\mathbb{N} = \{1, 2, 3, \ldots\}$

- Whole Numbers: $\mathbb{W} = \{0, 1, 2, 3, \ldots\}$

- Integers $\mathbb{Z} = \{\ldots, -3, -2, -1, 0,\ 1, 2,\ 3, \ldots\}$
  The numbers -3, -2, -1 are the opposite or negative of 3, 2, 1.

- Rational Numbers:

$$\mathbb{Q} = \{A\ rational\ number\ is\ a\ number\ that\ can\ be\ written\ as\ a\ quotient\ of\ two\ integers\ \frac{a}{b}, where\ b \neq 0\}$$

$\frac{a}{b}$ is a quotient of integers. e.g. $\frac{1}{5}$, $\frac{4}{3}$, 7, 3,...

7 is a rational number, since 7 can be written as a quotient of two integers $\frac{7}{1}$ or $\frac{14}{2}$, etc.

Proposition. *Every rational number can be written as an infinite repeating decimal. E.g.*

$$\frac{1}{3} = 0.333\cdots = 0.\bar{3}$$
$$\frac{1}{7} = 0.142857142857142857\cdots\cdots = 0.14\bar{2}857$$

The bar over 3, indicates that the number 3 repeats indefinitely.

- Irrational Numbers: $\mathbb{I} = \{$All the numbers that can not be written as a quotient of integers$\}$
  e.g. $\sqrt{2}$, $\sqrt{3}$, π, ...
  There are no two integers a and b, such that $\sqrt{2} = \frac{a}{b}$.

Definition. A perfect square is an integer whose square root is another integer. E.g.

$$\sqrt{1} = 1,\quad \sqrt{4} = 2,\quad \sqrt{9} = 3,\quad \sqrt{16} = 4,\quad \sqrt{25} = 5,\quad \sqrt{36} = 6,\quad \sqrt{49} = 7,\quad \sqrt{64} = 8,\quad \sqrt{81} = 9,\quad \sqrt{100} = 10$$

so,

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100, \ldots$$

are perfect squares.

Proposition. *A square root that is not equal to an integer is an irrational number.*

Proposition. *An irrational number is a number that has a nonterminating, nonrepeating decimal representation.*

Problem. *Is* $\sqrt{10}$ *a rational or irrational number?*

- all these numbers can be represented on a number line called the **Real number line**

Real numbers:

$$\begin{aligned} \mathbb{R} &= \{Rational\ numbers\} \cup \{Irrational\ numbers\} = \mathbb{Q} \cup \mathbb{I} \\ &= \{all\ the\ numbers\ that\ can\ be\ represented\ on\ a\ real\ number\ line\} \end{aligned}$$

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Problem. *What does mean that* π *is not a rational number.*

Problem. *Is it true that* $\pi = \dfrac{22}{7}$.

Problem. *Given*

$$A = \left\{3, -\frac{1}{3}, \sqrt{3}, \frac{1}{7}, 0.272727\cdots, \sqrt[3]{7}, -2, \frac{8}{7}, 3\frac{1}{4}, 0, \frac{1}{2}\right\}$$

Write the elements of each one of the following sets

1) $B = \{x : x \in A \text{ and } x \in I\}$

2) $C = \{x : x \in A \text{ and } x \in Q\}$

3) $D = \{x : x \in A \text{ and } x \in W\}$

4) $E = \{x : x \in A \text{ and } x \notin Q\}$

5) $F = \{x : x \in A \text{ and } x \in N\}$

7) $E = \{x : x \in A \text{ and } x \in Z\}$

Problem. *How many rational numbers there are between* $\dfrac{1}{3}$ *and* $\dfrac{1}{2}$

Problem. *Classify the following propositions as true or false.*

$W \subset N$

$W \subset Q$

$W \subseteq Q$

$I \subset Q$

$Q \cap Q' = \emptyset$

$\sqrt{2} \in Q$

$Q \cup Q' = R$

If $a \in Q$*, then* $a \in R$

If $a \in R$*, then* $a \in Q$

If $a \in I$*, then* $a \in Q$

If $a \in 0$*, then* $a \in Q$

$I \cup Q = R$

$0 \cup N = W$

$0 \cup N = W$

$0 \in W$

$0 \in W$

$\emptyset \subset N$

$(I \cup Q) \subset R$

$N \in R$

Problem. *Why the numbers 0.3, 3.61, 1, 1/7, 3.1416, 15%, 0.5% are rational numbers?*

Problem. *Find the decimal number equivalent to 7/8, 3/500,* $5\frac{2}{3}$*, 1/9, 7/11,* $14\frac{2}{5}$*%, 102%*

Problem. *Why is* $\sqrt{5}$ *an irrational number?*

Problem. *Find the equivalent fraction for* $0.03232\ldots$, $0.444\ldots$, $0.707070\ldots$

Problem. *Is the product of two even numbers even or odd?*

Problem. *Is the product of two odd numbers even or odd?*

Problem. *If* a^2 *is even, so what can you say about* a.

Problem. *If* a^2 *is odd, so what can you say about* a.

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Project #1.

Problem. 1. *Do a research about the number e (Euler number).*

Problem. 2. *Is it true that* $e = 2.7182818284$.

Problem. 3. *Given*

$$A = \left\{-5, -\frac{2}{3}, \sqrt{7}, \sqrt{16}, \frac{1}{3}, 0.5000\cdots, \sqrt[3]{8}, 5, \frac{4}{3}, 3\frac{1}{5}, -0, \sqrt{7}\right\}$$

Write the elements of each one of the following sets

1) $B = \{x : x \in A \text{ and } x \in I\}$

2) $C = \{x : x \in A \text{ and } x \in Q\}$

3) $D = \{x : x \in A \text{ and } x \in W\}$

4) $E = \{x : x \in A \text{ and } x \notin Q\}$

5) $F = \{x : x \in A \text{ and } x \in N\}$

7) $E = \{x : x \in A \text{ and } x \in Z\}$

Problem. 4. *Find 5 rational numbers between* $\frac{1}{3}$ *and* $\frac{1}{2}$

Problem. 5. *Classify the following propositions as true or false.*

1) $W \supset N$

2) $W \supset Q$

3) $W \supseteq Q$

4) $I \supset Q$

5) $Q \cup Q' = \emptyset$

6) $\sqrt{16} \in Q$

7) $Q \cap Q' = R$

8) *If* $a \in R$, *then* $a \in Q$

9) *If* $a \in I$, *then* $a \in Q$

10) *If* $a \in N$, *then* $a \in Q$

11) *If* $a \in 2$, *then* $a \in Q$

12) $I \cup Q = N \cup Q$

13) $0 \cup 3 = W$

14) $0 \cup Q = W$

15) $0 \in I$

17) $0 \in R$

18) $\emptyset \subset I$

19) $(I \cap Q) \subset R$

20) $N \in Q$

Problem. 6. *Why the numbers 0.5, 1.31, 4, 2/7, 2.718281, 5%, 0.02% are rational numbers?*

Problem. 7. *Find the decimal number equivalent to 1/371, 1/900,* $1\frac{3}{7}$*, 2/11, 1/93,* $14\frac{1}{5}$*%, 302%*

Problem. 8. *Why is* $\sqrt{93}$ *an irrational number?*

Problem. 9. *Find the equivalent fraction for* $0.0001522\cdots$, $0.999\cdots$, $0.151515\cdots$

Problem. 10. *Is the product of one even numbers times an odd number even or odd?*

Problem. 11. *Is the product of three odd numbers even or odd?*

Problem. 12. *If* a^3 *is even, so what can you say about* a.

Problem. 13. *If* a^3 *is odd, so what can you say about* a.

Help for Project #1.

Proposition. *Every rational number can be written as an infinite repeating decimal. E.g.*

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so,

$$1, \quad 4, \quad 9, \quad 16, \quad 25, \quad 36, \quad 49, \quad 64, \quad 81, \quad 100, \ldots$$

are perfect squares.

Proposition. *A square root that is not equal to an integer is an irrational number.*

Proposition. *An irrational number is a number that has a nonterminating, nonrepeating decimal representation.*

Problem. *Is $\sqrt{10}$ a rational or irrational number?*

$$9 < 10 < 16$$

calculating the square root to the previous expression,

$$3 < \sqrt{10} < 4$$

Since there is no integer between 3 and 4, $\sqrt{10}$ is not equal to an integer, so it is an irrational number.

Procedure. *You may obtain a rational number between other two taking the average of these two numbers:*

$$x < \frac{x+y}{2} < y$$

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Algorithms to calculate roots.

Procedure. *To obtain $\sqrt{2}$ select any value for x, then use the following formulas.*

$$y = \frac{x+2}{x+1}$$

then substitute y into:

$$z = \frac{y+2}{y+1}$$

then substitute z into:

$$t = \frac{z+2}{z+1}$$

and so on...

These different numbers x, y, z, t, approaches to $\sqrt{2}$.

Procedure. *To calculate the rth-root of N, we start taking an initial value x_0, and then using the following algorithm.*

$$x_1 = \frac{1}{r}\left[\frac{N}{x_0^{r-1}} + (r-1)x_0\right]$$
$$x_2 = \frac{1}{r}\left[\frac{N}{x_1^{r-1}} + (r-1)x_1\right]$$
$$x_3 = \frac{1}{r}\left[\frac{N}{x_2^{r-1}} + (r-1)x_2\right]$$
$$\vdots$$
$$x_{i+1} = \frac{1}{r}\left[\frac{N}{x_i^{r-1}} + (r-1)x_i\right]$$

This last iterative formula give us the aproximate value of the rth-root of N.

Example. Find the cubic root of 100.
Selecting a first aproximation, say, $x_0 = 4$, we have,

$$x_1 = \frac{1}{3}\left[\frac{100}{4^2} + (2)(4)\right] = 4.75$$
$$x_2 = \frac{1}{3}\left[\frac{100}{(4.75)^2} + (2)(4.75)\right] = 4.664$$
$$x_3 = \frac{1}{3}\left[\frac{100}{(4.664)^2} + (2)(4.664)\right] = 4.642$$
$$x_4 = \frac{1}{3}\left[\frac{100}{(4.642)^2} + (2)(4.642)\right] = 4.642$$

Therefore, $\sqrt[3]{100} = 4.642$.

Project #2.

Find an algoritm to calculate square roots different than the algorithms covered in class.

Project #3.

Obtain the aproximate value of π using the algorithms showed in Scientic American article pages 112B, 113, and 116. (upto $1/\alpha_3$)

Help for Project #3.

How to get two billions digits of Pi with a calculator*
Answer for algorithm in page 116.

Let

$y_0 = \sqrt{2} - 1 = 0.414213562$

and

$\alpha_0 = 6 - 4\sqrt{2} = 0.3431457$

$y_1 = \left[1 - \sqrt[4]{1 - y_0^4}\right] / \left[1 + \sqrt[4]{1 - y_0^4}\right]$

$y_1 = \left[1 - \sqrt[4]{1 - (0.414213562)^4}\right] / \left[1 + \sqrt[4]{1 - (0.414213562)^4}\right]$

$y_1 = 0.00373488546299$

$\alpha_1 = (1 + y_1)^4 \alpha_0 - 2^3 y_1 (1 + y_1 + y_1^2)$

$\alpha_1 = (1 + 0.00373488546299)^4\,(0.3431457) - 2^3(0.00373488546299)\left(1 + 0.00373488546299 + 0.00373488546299^2\right)$

$\alpha_1 = (1 + 0.00373488546299)^4\,(0.3431457) - 0.02999109545$

$\alpha_1 = 0.3483009311 - 0.02999109545$

$\alpha_1 = 0.31830983567$

$\pi \approx 1/\alpha_1$

$\pi \approx 1/0.31830983567$

$\pi \approx 3.14159315...$

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Properties of the Real Numbers.

Properties of the equality:

a) $a = a$ (reflexive)
b) If $a = b$, then $b = a$ (symmetric)
c) If $a = b$ and $b = c$, then $a = c$ (transitive)

Properties of Addition and Multiplication:

- Commutative Property:

  for every real numbers x, y we have
  $$x + y = y + x$$
  $$x \cdot y = y \cdot x$$

- Associative Property:

  for every real numbers x, y, z
  $$x + (y + z) = (x + y) + z$$
  $$x \cdot (y \cdot z) = (x \cdot y) \cdot z$$

- Identity Property:

  there exists a number 0, such that,
  for all real numbers x we have
  $$x + 0 = x$$
  and there exists a real number 1, such that,
  $1 \neq 0$ and for every real number x
  $$x \cdot 1 = x$$
  0 is called the identity element of addition, and 1 is called the identity element of multiplication.

- Inverse Property:

  for every real number x there exists a real number $-x$, such that
  $$x + (-x) = 0$$
  for every real number x, such that $x \neq 0$,
  there exists a real number $\frac{1}{x}$, such that
  $$x \cdot \frac{1}{x} = 1$$
  $-x$ is called the opposite of x or the additive inverse element, and $\frac{1}{x}$ the reciprocal of x or the multiplicative inverse element.

- Distributive Property:
  $$x \cdot (y + z) = x \cdot y + x \cdot z$$

Problem. *Complete each statement. Use a commutative property.*

1.- $1 + 9 =$

2.- $x + 9 =$

3.- $(2)(4) =$

4.- $2x =$

Problem. *Complete each statement. Use an associative property.*

1.- $(9 + 10) + 3 =$

2.- $5 + (2 + 8) =$

3.- $10 \cdot [8 \cdot 3] =$

Problem. *Decide whether each statement is an example of a commutative property, an associative property, or both.*

1.- $2(4 \cdot 6) = (2 \cdot 4)6$

2.- $(2 \cdot 4)6 = (4 \cdot 2)6$

3.- $(2 + 4) + 6 = 4 + (2 + 6)$

Problem. *Use an identity property to complete each statement.*

1.- $9 + 0 = \underline{\hspace{1.5em}}$

2.- $\underline{\hspace{1.5em}} + (-7) = -7$

3.- $\frac{1}{4} \cdot \underline{\hspace{1.5em}} = \frac{3}{12}$

4.- $\underline{\hspace{1.5em}} \cdot 1 = 5$

Problem. *Complete the statement so that they are examples of either an identity property or an inverse property. Tell which property is used.*

1.- $6 + \underline{\hspace{1.5em}} = 0$

2.- $\frac{4}{3} \cdot \underline{\hspace{1.5em}} = 1$

3.- $\frac{1}{9} \cdot \underline{\hspace{1.5em}} = 1$

4.- $275 + \underline{\hspace{1.5em}} = 275$

Problem. *Use distributive property to rewrite each expression.*

1.- $2(3+5)$

2.- $2(y+5)$

3.- $3 \cdot 5 + 3 \cdot 8$

4.- $7 \cdot 2 + 7 \cdot 9$

Problem. *Fill in the blanks with the correct responses.*

1.- *The identity element for addition is* ____________________.

2.- *The identity element for multiplication is* ____________________.

3.- *Every number has a(n)* ____________________ *inverse.*

4.- *Every number except* ____________________ *has a(n)* ____________________ *inverse.*

5.- *The sum of a number and its* ____________________ *is 0.*

6.- *The* ____________________ *of a number and its* ____________________ *is 1.*

7.- *The additive inverse of a is* ____________________.

8.- *The multiplicative inverse of a* $(a \neq 0)$ *is* ____________________.

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Definition. An Axiom p is a true statement.
A proposition or theorem p is a statement that can be classified as True or False but not both things at the same time. For all these theorems assume x, y, z, a, b, etc are real numbers.

Theorem 0.1. *If $x = y$ and c is any real number, then $x + c = y + c$.*

Proof.

$$\begin{aligned} &\text{by reflexivity property of equality, we have} \\ &x + c = x + c \\ &\text{using the hypothesis,} \\ &\text{and substitution} \\ &x + c = y + c \end{aligned}$$

□

Theorem 0.2. *If $x = y$ and c is any real number ($c \neq 0$), then $x \cdot c = y \cdot c$.*

Proof.

$$\begin{aligned} &\text{by reflexivity property of equality, we have} \\ &x \cdot c = x \cdot c \\ &\text{using the hypothesis,} \\ &\text{and substitution} \\ &x \cdot c = y \cdot c \end{aligned}$$

□

Theorem 0.3. *If $x + y = z + y$, then $x = z$*

Proof.

$$\begin{aligned} &\text{By the identity property of real numbers, we have} \\ &x + 0 = x \\ &\text{by the symmetric property of equality, we have} \\ &x = x + 0 \\ &\text{by the inverse property of real numbers, we have} \\ &x = x + (y + (-y)) \\ &\text{by the associative, we have} \\ &x = (x + y) + (-y) \\ &\text{by hypothesis, we have} \\ &x = (z + y) + (-y) \\ &\text{by the associative, we have} \\ &x = z + (y + (-y)) \\ &\text{by the inverse, we have} \\ &x = z + 0 \\ &\text{by the identity, we have} \\ &x = z \end{aligned}$$

□

Theorem 0.4. *If $x \cdot y = z \cdot y$, then $x = z$*

Proof.

By the identity property of real numbers, we have

$$x \cdot 1 = x$$

by the symmetric property of equality, we have

$$x = x \cdot 1$$

by the inverse property of real numbers, we have

$$x = x \cdot (y \cdot \frac{1}{y})$$

by the associative, we have

$$x = (x \cdot y) \cdot \frac{1}{y}$$

by hypothesis, we have

$$x = (z \cdot y) \cdot \frac{1}{y}$$

by the associative, we have

$$x = z \cdot (y \cdot \frac{1}{y})$$

by the inverse, we have

$$x = z \cdot 1$$

by the identity, we have

$$x = z$$

□

Theorem 0.5. *The identity element of addition is unique.*

Proof. Assume the contrary, i.e. there is another identity element, say b, such that $b \neq 0$. Then we have,

$$x + b = x$$

and

$$x + 0 = x$$

by substitution

$$x + b = x + 0$$

by theorem 0.3, we have

$b = 0$ but this is a contradiction "#"

therefore, our hypothesis is false, so the theorem is true, i.e. 0 is unique. □

Theorem 0.6. *The identity element of multiplication is unique.*

Proof. Assume the contrary, i.e. there is another identity element, say b, such that $b \neq 1$.
Then we have,

$$x \cdot b = x$$

and

$$x \cdot 1 = x$$

by substitution

$$x \cdot b = x \cdot 1$$

by theorem 0.4, we have

$$b = 1 \ \#$$

therefore, our hypothesis is false, so the theorem is true, i.e. 1 is unique. □

Theorem 0.7. *The additive inverse $-x$ of x is unique.*

Proof. Assume the contrary, i.e. there is another additive inverse of x, say b, such that $b \neq -x$.
Then we have,

$$x + b = 0$$

and

$$x + (-x) = 0$$

by substitution

$$x + b = x + (-x)$$

using theorem 0.3, we have

$$b = -x \ \#$$

therefore, our hypothesis is false, so the theorem is true, i.e. $-x$ is unique. □

Theorem 0.8. *The multiplicative inverse $\frac{1}{x}$ of x where $x \neq 0$ is unique.*

Proof. Assume the contrary, i.e. there is another multiplicative inverse of x, say b, such that $b \neq \frac{1}{x}$.
Then we have,

$$x \cdot b = 1$$

and

$$x \cdot \frac{1}{x} = 1$$

by substitution

$$x \cdot b = x \cdot \frac{1}{x}$$

by 0.4, we have

$$b = \frac{1}{x} \ \#$$

therefore, our hypothesis is false, so the theorem is true, i.e. $\frac{1}{x}$ is unique. □

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Project #4.

Problem. *Using the axioms for Real Numbers and the properties for equality, prove the following theorems:*

1)

Theorem 0.9. *For every x, we have that $x \cdot 0 = 0$.*

2)

Theorem 0.10. *For every x, we have that $-(-x) = x$.*

3)

Theorem 0.11. *If $x + y = 0$, then $y = -x$.*

4)

Theorem 0.12. $(-x) \cdot (-y) = x \cdot y$

Help for Project #4.

Theorem 0.13. *For every x, we have that $x \cdot 0 = 0$.*

Proof. We know that the opposite of $x \cdot 0$ is $-(x \cdot 0)$, set $c = -(x \cdot 0)$, then

By identity property we have that

$$0 + 0 = 0$$

by theorem 0.2, we have

$$x \cdot (0 + 0) = x \cdot 0$$

using the distributive property, we have

$$x \cdot 0 + x \cdot 0 = x \cdot 0$$

by theorem 0.1, we have

$$(x \cdot 0 + x \cdot 0) + c = x \cdot 0 + c$$

using the associative property on the left,

and the inverse property on the right side of the equality, we have

$$x \cdot 0 + (x \cdot 0 + c) = 0$$

by the inverse property, we have

$$x \cdot 0 + 0 = 0$$

by the identity property, we have

$$x \cdot 0 = 0$$

□

Theorem 0.14. *If $x + y = 0$, then $y = -x$.*

Proof.

by the identity property, we have

$$y = y + 0$$

by the inverse property, we have

$$y = y + (x + (-x))$$

by the associative property, we have

$$y = (y + x) + (-x)$$

by the commutative property, we have

$$y = (x + y) + (-x)$$

by the hypothesis of the theorem

$$y = 0 + (-x)$$

by the identity property, we have

$$y = -x$$

□

Theorem 0.15. *For every x, we have that $-(-x) = x$.*

Proof. We know that the opposite of $-x$ is $-(-x)$, set $c = -(-x)$, then

by the inverse property, we have

$$-x + c = 0$$

by theorem 0.1, we have
$$x + (-x + c) = x + 0$$
by the associative property, we have
$$(x + (-x)) + c = x + 0$$
by the inverse p. on the left side,
and the identity p. on the right side, we have
$$0 + c = x$$
by the identity property, we have
$$c = x$$
$$-(-x) = x$$

□

Theorem 0.16. $(-x) \cdot (-y) = x \cdot y$

Proof.

by the distributive property, we have
$$(-x) \cdot y + (-x) \cdot (-y) = (-x) \cdot (y + (-y))$$
by the inverse property, we have
$$(-x) \cdot y + (-x) \cdot (-y) = (-x) \cdot 0$$
by theorem 0.13, we have
$$(-x) \cdot y + (-x) \cdot (-y) = 0 \quad (1)$$
in the other hand,
by distributive property, we have
$$(-x) \cdot y + x \cdot y = ((-x) + x) \cdot y$$
by the inverse property, we have
$$(-x) \cdot y + x \cdot y = 0 \cdot y$$
by theorem 0.13, we have
$$(-x) \cdot y + x \cdot y = 0 \quad (2)$$
Using substitution with (1) and (2), we have
$$(-x) \cdot y + (-x) \cdot (-y) = (-x) \cdot y + x \cdot y$$
by theorem 0.3, we have
$$(-x) \cdot (-y) = x \cdot y$$

□

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Exponents, Parenthesis & the Order of Operations.

Definition. The expression a^n is called exponential expression, the number a is the base, and the number n is the exponent:

$$a^n = \overbrace{a \cdot a \cdot a \cdot a \cdots a}^{n-times}$$

Procedure. *Order of Operations.*

Step 1 *If possible simplify parentheses from inside to outside and above or below fractions bars.*
Step 2 *Apply all exponents.*
Step 3 *Do Multiplications or Divisions in the order in which they occur, working from left to right.*
Step 4 *Do Additions or Substractions in the order in which they occur, working from left to right.*

Problem. *Find the value of each exponential expression.*

1.- -6^2

2.- $(-6)^2$

3.- $\left(\frac{3}{4}\right)^2$

4.- $\left(\frac{1}{2}\right)^4$

5.- $(.4)^3$

Problem. *Find the value of each expression.*

1.- $\frac{2(7+8)+2}{3 \cdot 5 + 1}$.

2.- $5 + 3 \cdot [1 + (2 \cdot 3)] + 1$.

3.- $\frac{5 - [3(6 \div 3) - 2]}{5^2 - 4^2 \div 2}$.

4.- $3^2 + 3^2 + 2^3 + 2^3$.

5.- $3^2 \cdot 2^2 + 2^2 + 3^2$.

Problem. *Find the value of each expression if $x = 6$ and $y = 9$*

1.- $4x + 7y$

2.- $\frac{4x - 2y}{x + 1}$

3.- $2x^2 + y^2$

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Exponents, Parenthesis & the Order of Operations Cont...

As an example lets write the sum of the first 10 integers:

$$S = 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 10$$

or

$$S = [1 + 10] + [2 + 9] + [3 + 8] + [4 + 9] + [5 + 6]$$
$$S = 11 + 11 + 11 + 11 + 11$$
$$S = 5 \cdot 11$$
$$S = 55$$

Also, This sum may be written as:

$$1 + 2 + 3 + 4 + 5 + (10 - 4) + (10 - 3) + (10 - 2) + (10 - 1) + 10$$

Using the distributive property, we may add the first integer plus the last integer, the second with the one before the last, and so on:

$$S = [1 + 10] + [2 + (10 - 1)] + [3 + (10 - 2)] + [4 + (10 - 3)] + [5 + (10 - 4)]$$

or

$$S = [1 + 10] + [2 + (10 - 1)] + [3 + (10 - 2)] + [4 + (10 - 3)] + [\frac{10}{2} + (10 - 4)]$$

$$S = [1 + 10] + [2 + (10 - 1)] + [3 + (10 - 2)] + [4 + (10 - 3)] + [\frac{10}{2} + \frac{10}{2} + 1]$$

$$S = 11 + 11 + 11 + 11 + 11$$
$$S = 5 \cdot 11$$
$$S = 55$$

Now, the sum of the first n integers may be written as:

$$1 + 2 + 3 + 4 + 5 + \cdots + (n - 5) + (n - 4) + (n - 3) + (n - 2) + (n - 1) + n$$

Using the distributive property, we may add the first integer plus the last integer, the second with the one before the last, and so on:

$$S = [1 + n] + [2 + (n - 1)] + [3 + (n - 2)] + [4 + (n - 3)] + [5 + (n - 4)] + \cdots + [\frac{n}{2} + \frac{n}{2} + 1]$$

$$S = [n + 1] + [n + 1] + [n + 1] + [n + 1] + [n + 1] + \cdots + [n + 1]$$

we have $\frac{n}{2}$ pairs, therefore

$$S = \frac{n}{2}(n + 1)$$

Project #5.

Find a formula to calculate the sum of the first n^3 integers.

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The Summation Notation.

Definition. We express the sum $a_1 + a_2 + a_3 + \cdot + a_n$ as follows

$$a_1 + a_2 + a_3 + \cdots + a_n = \sum_{i=1}^{n} a_i$$

the summation notation $\sum_{i=1}^{n} a_i$ is something called the sigma notation. The index i is called the index of summaion, $i = 1$ is called the lower limit, n is called the upper limit, a_i is called the general term.

Proposition. *Rules for sums (for abreviation assume that $\sum = \sum_{i=1}^{n}$).*

1) $\sum c = nc$

2) $\sum(a_i + b_i) = \sum a_i + \sum b_i$

3) $\sum ca_i = c \sum a_i$

4) $\sum(ca_i + db_i) = c \sum a_i + d \sum b_i$

5) $n^b = \sum i^b - \sum(i-1)^b$

Theorem 0.17.

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Proof.

Use (5) from previous proposition with $b = 2$

$$n^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} (i-1)^2$$

$$n^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} (i^2 - 2i + 1)$$

using the rules for sums, we have

$$n^2 = \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$$

$$n^2 = \sum_{i=1}^{n} 2i - \sum_{i=1}^{n} 1$$

$$n^2 = 2\sum_{i=1}^{n} i - n$$

solving the previous equation for $\sum_{i=1}^{n} i$, we will have

$$\sum_{i=1}^{n} i = \frac{n^2 + n}{2}$$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

□

Theorem 0.18.

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

Proof.

Use (5) from previous proposition with $b = 3$

$$n^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} (i-1)^3$$

$$n^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} (i^3 + 3i^2(-1) + 3i(-1)^2 + (-1)^3)$$

$$n^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} (i^3 - 3i^2 + 3i - 1)$$

using the rules for sums, we have

$$n^3 = \sum_{i=1}^{n} i^3 - \sum_{i=1}^{n} i^3 + \sum_{i=1}^{n} 3i^2 - \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 1$$

$$n^3 = \sum_{i=1}^{n} 3i^2 - \sum_{i=1}^{n} 3i + \sum_{i=1}^{n} 1$$

$$n^3 = 3\sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i + n$$

we know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

substuting this expression into previous result, we have

$$n^3 = 3\sum_{i=1}^{n} i^2 - 3\frac{n(n+1)}{2} + n$$

now, solving this equation for $\sum_{i=1}^{n} i^2$, we will have

$$3\sum_{i=1}^{n} i^2 = n^3 - n + 3\frac{n(n+1)}{2}$$

then

$$\sum_{i=1}^{n} i^2 = \frac{n^3}{3} - \frac{n}{3} + \frac{n(n+1)}{2}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3}{6} - \frac{2n}{6} + \frac{3n(n+1)}{6}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 - 2n + 3n(n+1)}{6}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 - 2n + 3n^2 + 3n}{6}$$

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6}$$

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

□

Help for Project #5.

As you remember we already obtained a formula to calculate the sum of the first n consecutive integers:

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

i.e. $1+2+3+4+5+\cdots+n = \dfrac{n(n+1)}{2}$.

Also, we obtained a formula to calculate the sum of the first n consecutive square integers:

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

i.e. $1^2+2^2+3^2+4^2+5^2+\cdots+n^2 = \dfrac{n(2n+1)(n+1)}{6}$.

Now, using $n^b = \sum i^b - \sum (i-1)^b$ as in the last two cases, find a formula to calculate the sum of the first n consecuive cube integers:

$$\sum_{i=1}^{n} i^3$$

i.e. the sum of $1^3+2^3+3^3+4^3+5^3+\cdots+n^3 =?$.

Theorem 0.19.

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

Proof. Using (5) from the rules for summations with $b = 4$

$$n^4 = \sum_{i=1}^{n} i^4 - \sum_{i=1}^{n} (i-1)^4$$

$$n^4 = \sum_{i=1}^{n} i^4 - \sum_{i=1}^{n} (i^4 + \frac{4}{1!} i^3(-1) + \frac{12}{2!} i^2(-1)^2 + \frac{24}{3!} i(-1)^3 + (-1)^4)$$

$$n^4 = \sum_{i=1}^{n} i^4 - \sum_{i=1}^{n} (i^4 - 4i^3 + 6i^2 - 4i + 1)$$

using the rules for sums, we have

$$n^4 = \sum_{i=1}^{n} i^4 - \sum_{i=1}^{n} i^4 + 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1$$

$$n^4 = 4\sum_{i=1}^{n} i^3 - 6\sum_{i=1}^{n} i^2 + 4\sum_{i=1}^{n} i - n$$

now, solving this equation for $\sum_{i=1}^{n} i^3$, we will have

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + n + 6\sum_{i=1}^{n} i^2 - 4\sum_{i=1}^{n} i}{4}$$

we know that

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

and

$$\sum_{i=1}^{n} i^2 = \frac{n(2n+1)(n+1)}{6}$$

substuting these expressions into previous result, we have

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + n + n(2n^2 + 2n + n + 1) - 2n^2 - 2n}{4}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^4 + 2n^3 + n^2}{4}$$

then

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n^2 + 2n + 1)}{4}$$

$$\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$$

$\square$

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Order on $\mathbb{R}$.

Order on R: There is a subset $\mathbb{P}$ of $\mathbb{R}$, such that,

- Closure of $\mathbb{P}$: For every real numbers x, y such that $x, y \in \mathbb{P}$, then
$$x + y \in \mathbb{P}$$
and
$$x \cdot y \in \mathbb{P}$$
- Trichotomy axiom: For every real number x, one and only one of the following statements is true
$$x \in \mathbb{P}, \quad x = 0 \quad -x \in \mathbb{P}$$
The elements of $\mathbb{P}$ are called positive, and those such that $-x \in \mathbb{P}$ negative.

Definition. x is less than y, written $x < y$, if and only if
$$y - x \in \mathbb{P}$$
x is greater than x, written $y > x$, iff
$$x < y$$
The expressions $x \leq y$ or $y \geq x$ will mean that
$$x < y \quad \text{or} \quad x = y$$

Theorem 0.20. *If $x < y$ and $y < z$, then $x < z$*

Proof. Since $x < y$, then $y - x \in \mathbb{P}$. Since $y < z$, then $z - y \in \mathbb{P}$. Now by the closure of $\mathbb{P}$, we have
$$\begin{aligned}(y - x) + (z - y) &\in \mathbb{P}\\ (y + (-y)) + (z - x) &\in \mathbb{P}\\ 0 + z - x &\in \mathbb{P}\\ z - x &\in \mathbb{P}\\ \text{by definition of “} < \text{”}&\\ x &< z\end{aligned}$$
□

Theorem 0.21. *If $x < y$, and c is any real number, then $x + c < y + c$.*

Proof. Since $x < y$, then $y - x \in \mathbb{P}$, we have
$$\begin{aligned}(y - x) + 0 &\in \mathbb{P}\\ (y - x) + (c + (-c)) &\in \mathbb{P}\\ y + c - x - c &\in \mathbb{P}\\ (y + c) - (x + c) &\in \mathbb{P}\\ \text{by definition of “} < \text{”}&\\ x + c &< y + c\end{aligned}$$
□

Theorem 0.22. *If $x < y$, and $c \in \mathbb{P}$ (c is any positive real number), then $x \cdot c < y \cdot c$.*

Proof. Since $x < y$, then $y - x \in \mathbb{P}$, we also know that $c \in \mathbb{P}$, by the closure of $\mathbb{P}$, we have

$$c \cdot (y - x) \in \mathbb{P}$$
$$c \cdot y - c \cdot x \in \mathbb{P}$$
$$\text{by definition of “} < \text{”}$$
$$c \cdot x < c \cdot y$$
$$x \cdot c < y \cdot c$$

□

Theorem 0.23. *If $x < y$, and c negative, then $x \cdot c > y \cdot c$.*

Proof. Since $x < y$, then $y - x \in \mathbb{P}$, we also know that c is negative, therefore, $-c \in \mathbb{P}$, by the closure of $\mathbb{P}$, we have

$$-c \cdot (y - x) \in \mathbb{P}$$
$$-c \cdot y + c \cdot x \in \mathbb{P}$$
$$c \cdot x - c \cdot y \in \mathbb{P}$$
$$\text{by definition of “} < \text{”}$$
$$c \cdot y < c \cdot x$$
$$y \cdot c < x \cdot c$$
$$\text{by definition of “} > \text{”}$$
$$x \cdot c > y \cdot c$$

□

Theorem 0.24. *(Trichotomy) If $x, y \in \mathbb{R}$, then one and only one of the following properties hold.*

$$x < y \qquad y < x \qquad x = y$$

Proof. Since $x, y \in \mathbb{R}$, then $x - y \in \mathbb{R}$, By Trichotomy axiom we have that

$$x - y \in \mathbb{P} \quad \text{or} \quad -(x - y) \in \mathbb{P} \quad \text{or} \quad x - y = 0$$

by definition of “ $<$ ” and “ $>$ ”, we have

$$y < x \quad \text{or} \quad -x + y \in \mathbb{P} \quad \text{or} \quad x = y$$
$$y < x \quad \text{or} \quad y - x \in \mathbb{P}; \quad \text{or} \quad x = y$$
$$y < x \quad \text{or} \quad x < y \quad \text{or} \quad x = y$$
$$\text{i.e.}$$
$$x < y \quad \text{or} \quad y < x \quad \text{or} \quad x = y$$

□

Theorem 0.25. *$x < 0$ if and only if x is negative. $x > 0$ if and only if x is positive.*

Proof. Suppose $x < 0$, therefore $0 - x \in \mathbb{P}$, i.e. $-x \in \mathbb{P}$, therefore x is negative (by definition).
Assume that $x > 0$, therefore $0 < x$, or $x - 0 \in \mathbb{P}$, i.e. $x \in \mathbb{P}$, therefore x is positive (by definition). □

Theorem 0.26. $0 < 1$

Proof. By identity property of multiplication we know that $1 \neq 0$.
Now, suppose that $1 < 0$, then 1 is negative, so -1 is positive, i.e. $-1 \in \mathbb{P}$.
By closure of $\mathbb{P}$, we have

$$(-1) \cdot (-1) \in \mathbb{P}$$

so

$$1 \in \mathbb{P}$$

i.e. $1 > 0\#$
therefore, $0 < 1$. □

The Decimal System.

Definition. The decimal system is a place-value system of numeration whose base is 10.
In this system we use 10 Hindu-Arabic symbols

$$0\ 1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9$$

These symbols are referred to as digits.
Any number can be expressed uniquely as a sum of the terms, each of which is one of the basics symbols times a power of the base. The power of the base by which each of the basic symbols is to be multiplied is determined by its placement in relation to a reference point. In the decimal system this is called the *decimal point*. In any number, the first digit from left to right is called the first digit, and so on.

Names of the numbers.

Number	European Numerical System	American & Russian Numerical System
.001	1 thousandth	1 thousandth
.01	1 hundredth	1 hundredth
.1	1 tenth	1 tenth
10	1 ten	1 ten
10^2	1 hundred	1 hundred
10^3	1 thousand	1 thousand
10^6	1 million	1 million
10^9	1 thousand million	1 billion
10^{12}	1 billion	1 trillion
10^{15}	1 thousand billion	1 cuatrillion
10^{18}	1 trillion	1 quintillion
10^{21}	1 thousand trillion	1 sextillion
10^{24}	1 cuatrillion	1 septillion
10^{27}	1 thousand cuatrillion	1 octillion
10^{30}	1 quintillion	1 nonillion
...	...	...
10^{313}	...	1 centillion
10^{600}	1 centillion	...

E.g. 12345.12345 "twelve thousand three hundred forty five & twelve thousand three hundred forty five hundred thousandth (doce mil trescientos cuarenta y cinco milésimas)"

Problem. *Expanding and read the following numbers.*

1) $56,146,929 =$

2) $30,500,609,922 =$

3) $3,299.345 =$

4) $75,201,190.12345 =$

The Addition Algorithm.

Problem. *Using the algorithm of addition perform the following operations.*

1) $23 + 46 =$

2) $46 + 38 =$

Project #8 Part a.

Problem. *Using place-value format expand and read the following numbers.*

1) $929,826.123 =$

2) $597,609,922.123,456 =$

3) $785,299,546,100.123,456,789 =$

4) $100,001,001,001,001.123,456,789,111 =$

Project #8 Part b.

Problem. *Using the addition algorithm perform the following operations.*

1) $12 + 45 =$

2) $75 + 99 =$

3) $12 + 27 + 99 =$

<u>The Subtraction Algorithm.</u>

Problem. *Apply each one of the algorithms covered in class and the place value notation to perform the following operations.*

a) $49 - 15$

b) $73 - 29$

c) $132 - 43$

Other Number Systems.

MATHEMATICS 2303
Final Project

Instructor: Jorge Viramontes

Student Name:____**Call No:**____________________

===

This is a strictly personal project (no teams please), any kind of help is not allowed.
Project Due Date: Dec 8th, before 3:10pm at EDUC BLDG 211-F.

Problem. *Classify the following numbers as natural, integer, rational or irrational.*
Note: Some of them may fall in more than one category.

1) $\sqrt{144}$
2) $-\sqrt{144}$
3) $\sqrt{3}$
4) $\sqrt[5]{32}$
5) $\sqrt[3]{2/16}$

Problem. *Why is* $\sqrt{5}$ *an irrational number?*

Problem. *Write the number* 0.285714 *in the form* $\frac{a}{b}$.

Problem. *Prove the following theorem: If* x, y, z *are real numbers, and* $x = y$*, then* $x + z = y + z$.

Problem. *Using the order of operations (show every step), simplify*

$$\frac{2^3 + 3^3 + 3 \cdot 3}{3^2 + 3^2 + 2 \div 2}$$

Problem. *Find*

$$\sum_{i=1}^{n} i^4 = 1^4 + 2^4 + 3^4 + \cdots + n^4 = ?$$

Problem. *Prove that* $1 > 0$.

Problem. *Using place-value notation expand the following number and write its name.*

$$123, 456, 789, 123, 456, 789.123, 456, 789, 123$$

Problem. *Using place-value notation and the algorithm for addition calculate* $123 + 321 =$.

Problem. *Using place-value notation and each one of the algorithms for subtraction (we covered two different algorithms) calculate* $1234 - 999$.