

7.1 Graphing Linear Inequalities in Two Variables.

Definition. A linear inequality in two variables, is an inequality of the form $ax + by < c$ or $ax + by > c$, where a , b , or c are real numbers.

Procedure. *Graphing a Linear Inequality.*

- *Graph the boundary line (the line itself). Decide whether the line is part of the solution. Make a solid line if the inequality contains \leq or \geq , make a dash line if the inequality contains $<$ or $>$.*
- *Solve the inequality for y : Shade the region above the line if $y > mx + b$; shade the region below the line if $y < mx + b$.*
- *If you solve the inequality for x , instead of y , replace above and below with right and left .*

Problem. *Graph the linear inequalities.*

1) $2x + 5y \leq 10$.

2) $x - y \geq 4$.

3) $2x + 3y > 12$.

4) $x \geq 3$.

Definition. Two or more linear inequalities form a linear system of inequalities. The solution region is the region that satisfies all inequalities of the system, this region is called the **feasible region**.

Problem. *Solve the following systems of linear inequalities.*

•

$$\begin{aligned}x + y &\leq 6 \\ 2x + y &\geq 4\end{aligned}$$

•

$$\begin{aligned}x + 4y &\leq 8 \\ x - y &\geq 3 \\ x &\geq 0 \\ y &\geq 0\end{aligned}$$

- *Shell supplies two distributors located in Northwest. One distributor needs at least 3000 barrels of oil, and the other needs at least 5000 barrels. Shell can send out at most 10,000 barrels. Let x =the number of barrels of oil sent to distributor 1 and y =the number sent to distributor 2. Write a system of linear inequalities that describes all the conditions and graph the feasible region of the system.*

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 MATHEMATICS 2301

7.2 Linear Programming: The graphical method.

Many problems in bussines, science, and economics involve finding the **optimal value of a function** (for instance, the maximum profit, minimum cost, etc) subject to various **constraints** (such as transportation cost, enviromental protection laws, etc.). **Linear Programming** deals with such situations in which the function to be optimized, called the **objective function**, is linear and the constrains are given by **linear inequalities**.

Problem. Find the maximum and minimum values of the objective function $z = 2x + 5y$, subject to the following constraints (consider $z = 0$, $z = 5$, $z = 10$, and $z = 15$).

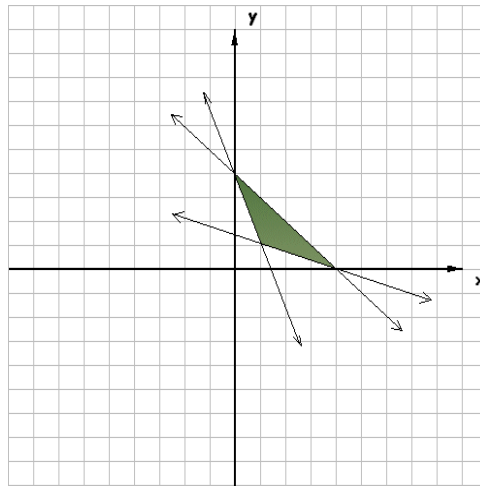
$$\begin{aligned} 3x + 2y &\leq 6 \\ -2x + 4y &\leq 8 \\ x + y &\geq 1 \\ x, y &\geq 0 \end{aligned}$$

Remark. Point such as A in the previous problem is called corner point. A **corner point** is a point in the feasible region where the boundary lines of two constraints cross. The feasible region is **bounded**, because the region is enclosed by boundary lines on all sides. Linear programming problems with bounded regions always have solutions. But if we did not include the constraint $3x + 2y \leq 6$, the feasible region would be **unbounded**, and there would be no way to *maximize* the value of the **objective function**.

Proposition. *Corner Point Theorem.* If the feasible region is bounded, then the objective function has both a maximum and a minimum value and each occurs at one or more corner points. If the feasible region is unbounded, the objective function may not have a maximum or minimum. But if a maximum or minimum value exists, it will occur at once or more corner points.

Problem. Use the corner point theorem.

- Identify the corner points in the graph.
- Which corner point would minimize $z = 2x + 3y$?

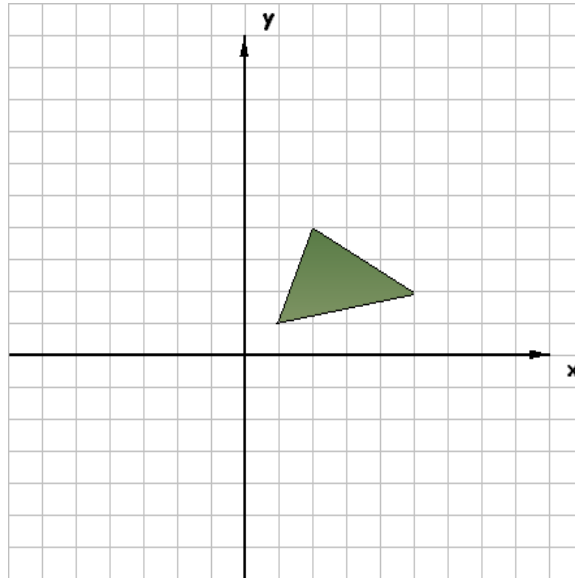


Procedure. *Solving a Linear Programming Problem Graphically.*

- Write the objective function and all necessary constraints.
- Graph the feasible region.
- Determine the coordinates of each of the corner points.
- Find the value of the objective function at each corner point.
- If the feasible region is bounded, the solution is given by the corner point producing the optimum value of the objective function.
- If the feasible region is an unbounded region in the first quadrant and both coefficients of the objective function are positive, then the minimum value of the objective function occurs at a corner point and there is no maximum value.

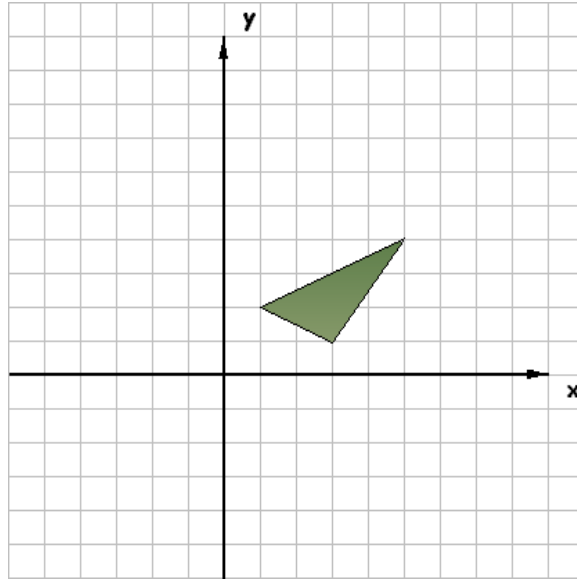
Problem. *Use the region of feasible solutions in the sketch to find the following.*

- The values of x and y that maximize $z = 2x - y$.*
- The maximum value of $z = 2x - y$.*
- The values of x and y that minimize $z = 4x + 3y$.*
- The minimum value of $z = 4x + 3y$.*

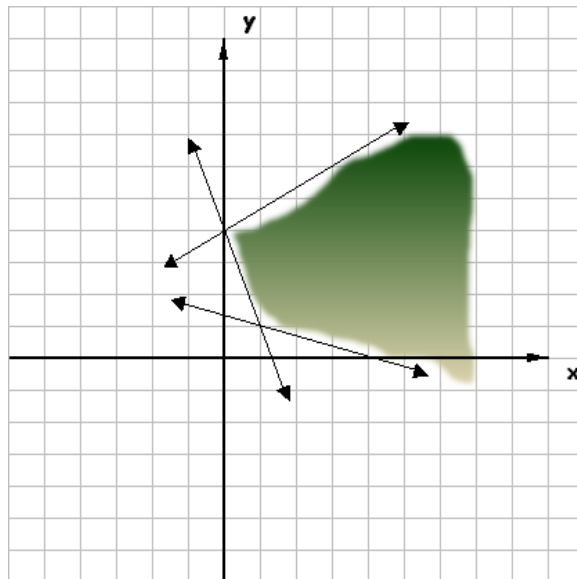


Problem. The sketch shows a feasible region. Let $z = x + 3y$. Use the sketch to find the values of x and y that

- minimize z ;
- maximize z .



Problem. The sketch below shows a region of feasible solutions. From the sketch decide what ordered pair would minimize $z = 2x + 4y$.



Instructor: Jorge R. Viramontes
MATHEMATICS 2301

7.3 Applications of Linear Programming.

Problem. *A Club member raises geese and pigs. She wants to raise no more than 16 animals including no more than 10 geese. She spends \$15 to raise a goose and \$45 to raise a pig, and has \$540 available for this project. Find the maximum profit she can make if each goose produces a profit of \$7 and each pig a profit of \$20.*

Problem. *An office manager needs to purchase new filing cabinets. He knows that Ace cabinets cost \$40 each, require 6 square feet of floor space, and hold 8 cubic feet of files. On the other hand, each Excello cabinet costs \$80, requires 8 square feet of floor space, and holds 12 cubic feet. His budget permits him to spend no more than \$560 on files, while the office has room for no more than 72 square feet of cabinets. The manager desires the greatest storage capacity within the limitations imposed by funds and space. How many of each type of cabinet should he buy?*

Problem. *A popular cereal combines oats and corn. At least 27 tons of cereal is to be made. For the best flavor, the amount of corn should be no more than twice the amount of oats. Corn costs \$200 per ton and oats cost \$300 per ton. How much of each grain should be used to minimize the cost?*

- Make a chart to organize the information given in the problem?*
- Write an equation for the objective function.*
- Write four inequalities for the constraints.*

Problem. *Certain laboratory animals must have at least 30 grams of protein and at least 20 grams of fat per feeding period. These nutrients come from food A, which cost \$0.18 per unit and supplies 2 grams of protein and 4 of fat, and food B, with 6 grams of protein and 2 of fat, costing \$0.12 per unit. Food B is bought under a long-term contract requiring that at least 2 units of B be used per serving. How much of each food must be bought to produce minimum cost per serving?*

- Graph the feasible region and find the corner points.*
- Determine the minimum value of the objective function and the point where it occurs.*
- Is there a maximum cost?*

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7.4 The Simplex Method: Maximization.

For linear programming problems with more than two variables or with two variables and many constraints, the graphical method is usually too inefficient, so the **simplex method is used**.

Definition. Standard Maximum Form. A linear programming problem is in **standard maximum form** if

- 1.- The objective function is to be maximized;
- 2.- All variables are nonnegative ($x_i \geq 0$, $i = 1, 2, 3, \dots$);
- 3.- All constraints involve \leq ;
- 4.- The constants on the right side in the constraints are all nonnegative ($b \geq 0$).

Procedure. *Simplex Method.*

- 1.- *Determine the objective function.*
- 2.- *Write all necessary constraints.*
- 3.- *Convert each constraint into an equation by adding slack variables.*
- 4.- *Set up the initial simplex tableau.*
- 5.- *Locate the most negative indicator. If there are two such indicators, choose one. This indicator determines the pivot column.*
- 6.- *Use the positive entries in the pivot column to form the quotients necessary for determining the pivot.*
(The most negative indicator identifies the variable that is to be eliminated from all but one of the equations).
If there are no positive entries in the pivot column, no maximum solution exists. If two quotients are equally the smallest, let either determine the pivot (It may be that the first choice of a pivot does not produce a solution. In that case try the other choice). If the column contains zeroes or negative entries, take only the positive entries in the pivot column.
- 7.- *Multiply every entry in the pivot row by the reciprocal of the pivot to change the pivot to 1. Then use row operations to change all other entries in the pivot column to 0 by adding suitable multiples of the pivot row to the other rows.*
- 8.- *If the indicators are all positive or 0, this is the final tableau. If not, go back to Step 5 above and repeat the process until a tableau with no negative indicators is obtained.*
- 9.- *Determine the basic and nonbasic variables and read the solution from the final tableau. The maximum value of the objective function is the number in the lower right-hand corner of the final tableau.*

Problem. *Solve the following linear programming problem.*

$$\begin{aligned}
 &\text{Maximize } z = 2x_1 + 3x_2 + x_3 \\
 &\text{Subject to: } x_1 + x_2 + 4x_3 \leq 100 \\
 &\quad \quad \quad x_1 + 2x_2 + x_3 \leq 150 \\
 &\quad \quad \quad 3x_1 + 2x_2 + x_3 \leq 320 \\
 &\text{with: } x_1 \geq 0, x_2 \geq 0, x_3 \geq 0.
 \end{aligned}$$

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MATHEMATICS 2301

7.5 Maximization Applications.

Problem. *A cat breeder has the following amounts of cat food: 90 units of tuna, 80 units of liver, and 50 units of chicken. To raise a Siamese cat, the breeder must use 2 units of tuna, 1 of liver, and 1 of chicken per day, while raising a Persian cat requires 1, 2, and 1 units, respectively, per day. If a Siamese cat sells for \$12 while a Persian cat sells for \$10, how many of each should be raised in order to obtain maximum gross income? What is the maximum gross income?*

Problem. *Super Souvenir Company makes paper weights, plaques, and ornaments. Each paperweight requires 8 units of plastic, 3 units of metal, and 2 units of paint. Each plaque requires 4 units of plastic, and 1 unit each of metal and paint. Each ornament requires 2 units each of plastic and metal, and 1 unit of paint. They make a profit of \$3 on each paperweight and each ornament, and \$4 on each plaque. If 36 units of plastic, 24 units of metal, and 30 units of paint are available today, how many of each kind of souvenir should be made in order to maximize profit?*

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 MATHEMATICS 2301

11.1 Limits.

We have often dealt with a problem like this: Find the value $f(x)$ of the function f at $x = a$. The underlying idea of “limit”, however, is to examine what the function does near $x = a$ (in a neighborhood of a), rather than what it does at $x = a$.

Definition. Limit of a Function.

Let f be a function and let a and L be real numbers. Assume that $f(x)$ is defined for all x near $x = a$. Suppose that

as x takes values very close (but not equal) to a (on both sides of a), the corresponding values of $f(x)$ are very close (and possibly equal) to L ; and that the values of $f(x)$ can be made arbitrarily close to L for all values of x that are close enough to a (As x gets closer & closer to “ a ”, from either side, the corresponding $f(x)$ values gets closer & closer to L),

Then the number L is the **limit** of the values of $f(x)$ as x approaches a , which is written

$$\boxed{\lim_{x \rightarrow a} f(x) = L}$$

Remark. If the limit of a function $f(x)$ as x approaches a exists, this limit need not be equal to the number $f(a)$. In fact, $f(a)$ may not be defined.

Problem. Analyze the function f defined by $f(x) = x + 3$.

- at $x = 3$;
- in a neighborhood of 3.

Problem. Analyze the function f defined by $f(x) = \frac{x^2 - 9}{x - 3}$.

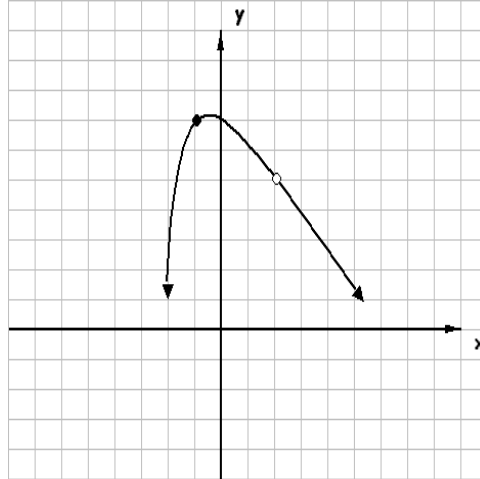
- at $x = 3$;
- in a neighborhood of 3.

Problem. Analyze the function f defined by $f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$

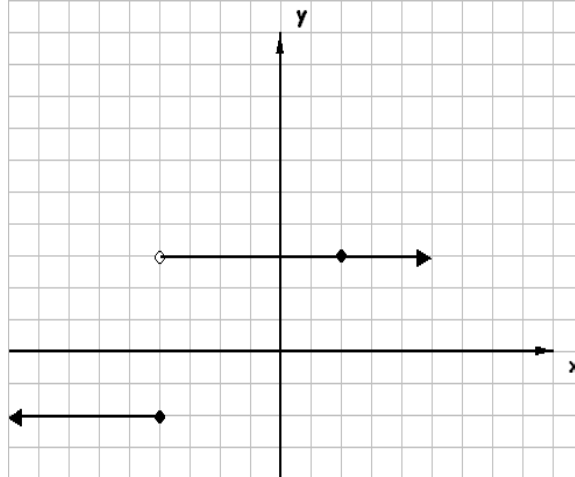
- at $x = 0$,
- in a neighborhood of 0.

Problem. Using each graph determine the value of the limit.

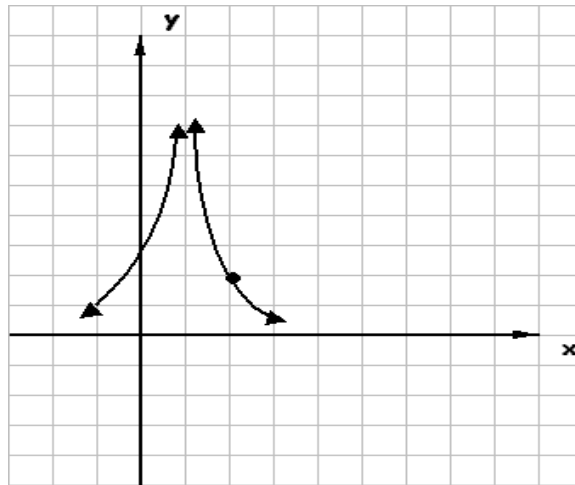
1.- $\lim_{x \rightarrow 2} f(x)$ and $\lim_{x \rightarrow -1} f(x)$.



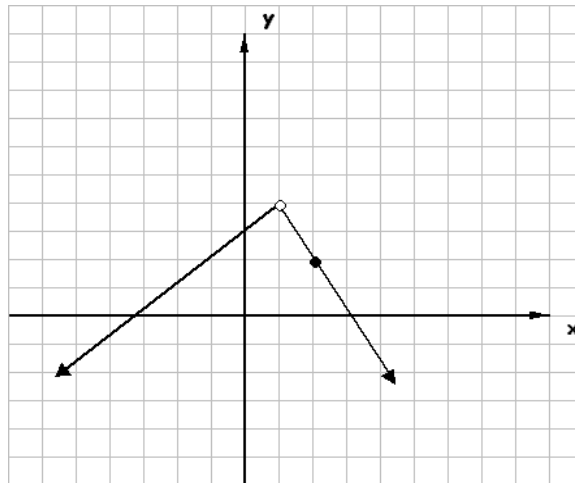
2.- $\lim_{x \rightarrow -4} f(x)$ and $\lim_{x \rightarrow 2} f(x)$.



3.- $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow 2} f(x)$.



4.- $\lim_{x \rightarrow 1} f(x)$ and $\lim_{x \rightarrow 2} f(x)$.



Proposition. Properties of Limits.

Let a , k , A , and B be real numbers, and let f and g be functions such that

$$\lim_{x \rightarrow a} f(x) = A \quad \text{and} \quad \lim_{x \rightarrow a} g(x) = B$$

then

1.

$$\lim_{x \rightarrow a} k = k \quad (\text{for any constant } k)$$

2.

$$\lim_{x \rightarrow a} x = a \quad (\text{for any real number } a)$$

3.

$$\lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x) = A \pm B$$

4.

$$\lim_{x \rightarrow a} [f(x) \cdot g(x)] = \lim_{x \rightarrow a} f(x) \cdot \lim_{x \rightarrow a} g(x) = A \cdot B$$

5.

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)} = \frac{A}{B} \quad (B \neq 0)$$

6. For any real number r for which A^r exists,

$$\lim_{x \rightarrow a} [f(x)]^r = [\lim_{x \rightarrow a} f(x)]^r = A^r$$

Proposition. Polynomials Limits.

If $f(x)$ is a polynomial function and a is a real number, then

$$\lim_{x \rightarrow a} f(x) = f(a)$$

Proposition. Limit Theorem.

If f and g are functions that have limits as x approaches a , and $f(x) = g(x)$ for all $x \neq a$, then

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} g(x)$$

Proposition. Existence of Limits.

The limit of a function f as x approaches a may fail to exist if

- $f(x)$ becomes infinitely large in absolute value as x approaches a from either side; or
- $f(x)$ gets closer and closer to a number L_1 as x approaches a from the left, but $f(x)$ gets closer and closer to a different number L_2 as x approaches a from the right.

Problem. Suppose $\lim_{x \rightarrow 4} f(x) = 16$ and $\lim_{x \rightarrow 4} g(x) = 8$. Use the limit properties to find the following limits.

1.- $\lim_{x \rightarrow 4} [f(x) + g(x)].$

2.- $\lim_{x \rightarrow 4} \frac{f(x) + g(x)}{2g(x)}.$

3.- $\lim_{x \rightarrow 4} [g(x)]^3$

Problem. Graph the function f whose rule is

$$f(x) = \begin{cases} 3 - x & \text{if } x < -2 \\ x + 2 & \text{if } -2 \leq x < 2 \\ 1 & \text{if } x \geq 2 \end{cases}$$

Use the previous graph to find $\lim_{x \rightarrow -2} f(x)$, $\lim_{x \rightarrow 1} f(x)$, and $\lim_{x \rightarrow 2} f(x)$

Problem. Use algebra and the properties of limits as needed to find the following limits.

1.- $\lim_{x \rightarrow 1} (4x^3 - x^2 + 3x - 1)$.

2.- $\lim_{x \rightarrow -2} \frac{2x + 1}{3x - 4}$.

3.- $\lim_{x \rightarrow -2} \frac{x^2 - 4}{x + 2}$.

4.- $\lim_{x \rightarrow 5} \frac{x^2 - 3x - 10}{x - 5}$.

5.- $\lim_{x \rightarrow 3} \sqrt{x^2 - 5}$.

6.- $\lim_{x \rightarrow 0} \frac{[-1/(x + 2)] + 1/2}{x}$.

7.- $\lim_{x \rightarrow 36} \frac{\sqrt{x} - 6}{x - 36}$.

8.- $\lim_{x \rightarrow 8} \frac{x - 8}{\sqrt{x} - \sqrt{8}}$.

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 MATHEMATICS 2301

11.2 Rate of Change.

Definition. Let f be a function, the average rate of change for $f(x)$ from $x = a$ to $x = b$ is

$$\frac{f(b) - f(a)}{b - a},$$

Definition. Let f be a function, the instantaneous rate of change for $f(x)$ at x_0 , where $h = x - x_0$ is

$$\lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists.

Definition. If an object moves along a straight line, with position $s(t)$ at time t and $h = t - t_0$, then the velocity of the object at t_0 is

$$v = \lim_{t \rightarrow t_0} \frac{s(t) - s(t_0)}{t - t_0} = \lim_{h \rightarrow 0} \frac{s(t_0 + h) - s(t_0)}{h},$$

provided this limit exists.

Definition. The rate of change of the cost function is called the marginal cost. Similarly, marginal revenue and marginal profit are the rates of change of the revenue and the profit functions, respectively.

Problem. Find the average rate of change for the following functions over the given closed intervals.

1.- $f(x) = -3x^3 + 2x^2 - 4x + 1$ between $x = 0$ and $x = 1$.

2.- $f(x) = \frac{1}{x - 1}$ between $x = -2$ and $x = 0$.

Problem. A car is moving along a straight road. At time t seconds the distance of the car (in feet) from the starting points is $s(t) = 2t^2$.

1.- Find the average speed of the car during the first 5 seconds, and 20 seconds.

2.- Find the instantaneous velocity (speed) of the car at $t = 5$, and $t = 20$. (for each t use a table, then use the definition of limit)

Problem. Suppose customers in a hardware store are willing to buy $N(p)$ boxes of nails at p dollars per box, as given by $N(p) = 80 - 5p^2$, $1 \leq p \leq 4$.

1.- Find the average rate of change of the demand for a change in price from 2 to 3.

2.- Find the instantaneous rate of change of demand when the price is 2.

3.- Find the instantaneous rate of change of the demand when the price is 3.

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11.3 Tangent Lines and Derivatives.

Definition. The **tangent** line to the graph of $y = f(x)$ at the point $(x_0, f(x_0))$ is the line through this point having slope

$$m_{\text{tangent}} = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h},$$

provided this limit exists. If this limit does not exist, then there is no tangent at the point.

Definition. The **derivative** of the function f is the function denoted f' whose value at the number x is defined to be the number

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h},$$

provided this limit exists.

Remark. Derivatives also do not exist at “corners” or “sharp points” on graphs (the derivative of $f(x) = |x|$ does not exist at $x_0 = 0$).

Procedure. *Finding $f'(x)$ from the definition of the derivative.*

- 1.- Find $f(x + h)$.
- 2.- Find $f(x + h) - f(x)$.
- 3.- Divide by h to get $\frac{f(x + h) - f(x)}{h}$.
- 4.- Let $h \rightarrow 0$; $f'(x) = \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$ if this limit exists.

Problem. *Find the slope of the tangent line to each of the following curves when x has the given value.*

- 1.- $f(x) = 6x^2 - 4x$; $x = -2$.
- 2.- $f(x) = \frac{6}{x}$; $x = -1$.
- 3.- $f(x) = -3\sqrt{x}$; $x = 1$.

Problem. *Find the equation of the tangent line in standard form to each of these curves at the given point.*

- 1.- $f(x) = 6 - x^2$; $x = -2$.
- 2.- $f(x) = -\frac{3}{x+1}$; $x = 1$.

Problem. *Draw the graph of a function f which has the following properties. (Many correct answers are possible).*

- $f'(x) > 0$ for $x < -2$.
- $f'(x) = 0$ at $x = -2$.
- $f'(x) < 0$ for $-2 < x < 3$.
- $f'(x) = 0$ at $x = 3$.
- $f'(x) > 0$ for $x > 3$.

Problem. Find $f'(x)$ for each function.

1.- $f(x) = 8x + 6$.

2.- $f(x) = 6x^2 - 4x$.

3.- $f(x) = \frac{6}{x}$.

4.- $f(x) = \sqrt{x}$.

5.- $f(x) = |x|$.

Definition. The derivative of the cost $C'(x)$ (instantaneous rate of change of the cost function) is called the marginal cost. Marginal cost can be thought of as the cost of producing one more item after x units have been produced. Similarly, marginal revenue and marginal profit are the rates of change of the revenue and the profit functions, respectively.

Problem. The cost of producing x tacos is $C(x) = 1000 + .24x^2$, $0 \leq x \leq 30,000$.

1.- Find the marginal cost, $C'(x)$.

2.- Find and interpret $C'(100)$.

3.- Find the exact cost to produce 101 st taco.

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11.4 Techniques For Finding Derivatives.

Definition. Notations for the Derivatives.

The derivatives of the function $y = f(x)$ may be denoted in any of the following ways.

$$f'(x), \quad y', \quad \frac{dy}{dx}, \quad \frac{d f(x)}{dx}, \quad D_x y, \quad D_x[f(x)]$$

Proposition. *Formulas to calculate derivatives. Let $u = u(x)$, and $v = v(x)$, and suppose that v' and u' exist, then*

(1) If $y = c$, where c is a real number then $\frac{d y}{d x} = \frac{d c}{d x} = 0$.

(2) If $y = x$, then $\frac{d y}{d x} = \frac{d x}{d x} = 1$.

(3) If $y = x^n$, for any nonzero real number n , then $\frac{d y}{d x} = \frac{d x^n}{d x} = n x^{n-1}$.

(4) Let c be a real number, then the derivate of $y = c \cdot v$ is $\frac{d y}{d x} = \frac{d (c \cdot v)}{d x} = c \cdot \frac{d v}{d x}$

(5) If $y = u \pm v$, then $\frac{d y}{d x} = \frac{d (u \pm v)}{d x} = \frac{d u}{d x} \pm \frac{d v}{d x}$

Definition. If $C(x)$ is the cost function, then **the marginal cost** (rate of change of the cost) is given by the derivative $C'(x)$.

$C'(x) \approx$ cost of making one more item after x have been made.

The marginal revenue $R'(x)$ and marginal profit $P'(x)$ are interpreted similarly.

Definition. The **demand function** $p(x)$, relates the number of units x of an item that consumers are willing to purchase at the price p . The total revenue $R(x)$ is related to the price per unit and the amount demanded (or sold) by the equation

$$R(x) = xp = x \cdot p(x).$$

Problem. Find the derivatives of these equations.

1) $f(x) = 8x^2 + x - 12$.

2) $y = 5x^6 - 3x^4 + x^3 - 3x + 9$.

3) $y = -100\sqrt{x} - 11x^{2/3}$.

4) $y = x^{-5} - x^{-2} + 5x^{-1}$.

5) $y = \frac{3 + x - 7x^4}{x^6}$.

Problem. What is the y -intercept of the tangent line to the graph of $y = x^3 - 2x^2 - 3$ at the point where $x = 1$.

Problem. The profit in dollars from the sale of x expensive cassette recorders is

$$P(x) = x^3 - 5x^2 + 7x + 10$$

Find the marginal profit for $x = 4$

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MATHEMATICS 2301

11.5 Derivatives of Products and Quotients.

Proposition. Let $u = u(x)$, and $v = v(x)$, and suppose that u' and v' exists, if $y = u \cdot v$, then

$$(6) \quad \boxed{\frac{dy}{dx} = \frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}}$$

Proposition. If $y = \frac{u}{v}$, if all indicated derivatives exists, and if $v \neq 0$, then

$$(7) \quad \boxed{\frac{dy}{dx} = \frac{d(u/v)}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}}$$

Problem. Find the derivative of the following equations.

1) $y = (2x^2 + 3)(4x + 5)$.

2) $y = (3x^3 + x^2)^2$.

3) $f(x) = \frac{7x + 1}{3x + 8}$.

4) $y = \frac{5x + 6}{\sqrt{x}}$.

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11.6 The Chain Rule.

Definition. Composite Function.

Let f and g be functions. The **composite function**, or **composition** of f and g is the function whose values are given by $f(g(x))$ for all x in the domain of g such that $g(x)$ is in the domain of f .

Definition. Chain Rule.

If y is a function of u , say $y = y(u)$ and if u is a function of x , say $u = u(x)$, then

$$\frac{dy[u(x)]}{dx} = y'[u(x)] \cdot u'(x)$$

i.e.

$$8) \quad \boxed{\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}}$$

Proposition. Generalized Power Rule.

Let u be a function of x , and let $y = u^n$, for any real number n . Then

$$(9) \quad \boxed{\frac{dy}{dx} = \frac{du^n}{du} \cdot \frac{du}{dx} = n \cdot u^{n-1} \cdot \frac{du}{dx}}$$

Problem. Let $f(x) = 2x^2 + 3x$ and $g(x) = 4x - 1$. Find $f(g(3))$.

Problem. Using the derivative for composition function find the derivatives.

$$1) \quad y = (x^2 + 2)^{1/2}.$$

$$2) \quad y = (x^2 + 5x)^{1/3} - 2(x^2 + 5x)^{2/3} + 7.$$

Definition. The **marginal revenue product** is the change in revenue when a manufacturer hires an additional employee. Suppose that R is the total revenue from the daily production of x units and p is the price per unit, then $R = px$. x depends on the number of employees, so the demand function p , then

$$\frac{dR}{dn} = \frac{d(px)}{dn} = p \cdot \frac{dx}{dn} + x \cdot \frac{dp}{dn}$$

now by the chain rule $\frac{dp}{dn} = \frac{dp}{dx} \cdot \frac{dx}{dn}$, we have $\frac{dR}{dn} = \left(p + x \cdot \frac{dp}{dx} \right) \frac{dx}{dn}$.

Problem. Find the marginal revenue product (in dollars) when $n = 20$ if the demand function is $p = 600/\sqrt{x}$ and $x = 5n$.

Instructor: Jorge R. Viramontes
 MATHEMATICS 2301

11.7 Derivatives of Exponential and Logarithmic Functions.

Proposition. Derivative of $y = e^u(x)$. If $u = u(x)$, and if $y = e^{u(x)}$, then

$$(10) \quad \boxed{\frac{d y}{d x} = \frac{d e^u}{d x} = e^u \cdot \frac{d u}{d x}}$$

Proposition. Derivative of $y = \ln |u|$ or $y = \ln u$ for $u > 0$.

$$(11) \quad \boxed{\frac{d y}{d x} = \frac{d \ln u}{d x} = \frac{1}{u} \cdot \frac{d u}{d x}}$$

Remark. Remember that $y = \ln |x|$ means $x = e^y$.

Problem. Find the derivative of the following equations.

1) $y = e^{-4x}$.

2) $y = 4e^{2x^2-4}$.

3) $y = \ln \left(\frac{6-x}{3x+5} \right)$.

4) $y = \frac{3 \ln |x|}{3x+4}$.

5) $y = e^{x^3} \ln |x|$.

11.8 Continuity and Differentiability.

Definition. A function f is **continuous** at $x = c$ if

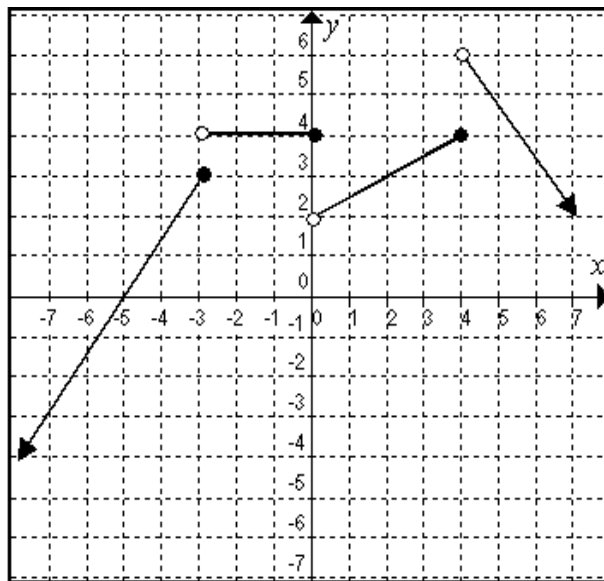
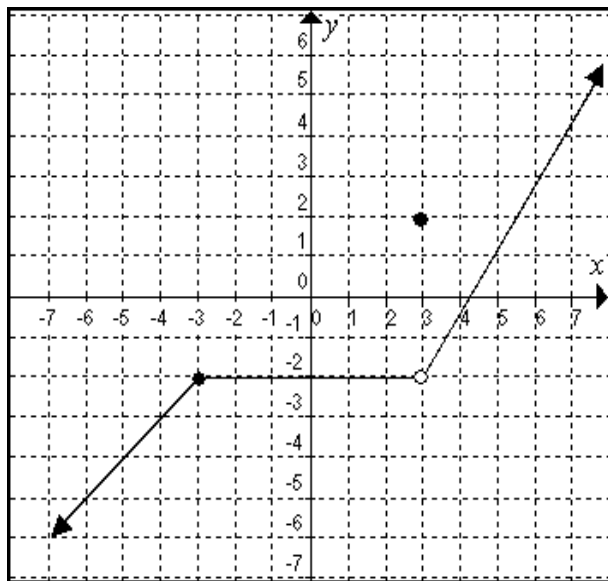
- a) $f(c)$ is defined;
- b) $\lim_{x \rightarrow c} f(x)$ exists;
- c) $\lim_{x \rightarrow c} f(x) = f(c)$.

If f is not continuous at $x = c$, it is discontinuous there.

If a function is continuous at each point of an open interval, it is said to be continuous on the open interval.

Proposition. If the derivative of a function exists at a point, then the function is continuous at that point. However, a function may be continuous at a point and not have a derivative there.

Problem. Find all points of discontinuity for the functions whose graphs are shown next.



Problem. Are the following functions continuous at the given values of x ?

1) $y = \frac{5}{x+5}$; $x = -5$, $x = 5$.

2)
$$g(x) = \begin{cases} e^x & \text{if } x < 0 \\ x + 1 & \text{if } 0 \leq x \leq 3; x = 0, x = 3. \\ 2x - 3 & \text{if } x > 3 \end{cases}$$

12.1 Derivative and Graphs & 12.4 Curve Sketching.

Proposition. Suppose a function f has a derivative at each point in an open interval:

- 1) If $f'(x) > 0$ for each x in the interval, f is increasing on the interval.
- 2) If $f'(x) < 0$ for each x in the interval, f is decreasing on the interval.
- 3) If $f'(x) = 0$ for each x in the interval, f is constant on the interval.

Definition. If f is a function, then a number c for which $f(c)$ is defined and either $f'(c) = 0$ or $f'(c)$ doesn't exist is called a **critical number** of f . The corresponding point $(c, f(c))$ on the graph of f is called a **critical point**.

Procedure. Determining when a function f is increasing or decreasing.

Step 1.- Compute the derivative f' .

Step 2.- Find the critical numbers of f .

Step 3.- Solve the inequalities $f'(x) > 0$ and $f'(x) < 0$ by testing a number in each of the intervals determined by the critical numbers.

The solutions of $f'(x) > 0$ are intervals on which f is increasing and the solutions of $f'(x) < 0$ are intervals on which f is decreasing.

Definition. Let c be a number in the domain of a function f .

- 1) f has a **local maximum** at c if $f(x) \leq f(c)$ for all x near c .
- 2) f has a **local minimum** at c if $f(x) \geq f(c)$ for all x near c .

The function f is said to have a **local extremum** at c if it has a local maximum or minimum there.

If f has a local extremum at $x = c$, then c is a critical number of f .

Proposition. First Derivative Test.

Assume that $a < c < b$ and that c is the only critical number for a function f in the interval $[a, b]$. Assume that f is differentiable for all x in $[a, b]$, except possibly at $x = c$.

- 1) If $f'(a) > 0$ and $f'(b) < 0$, then there is a local maximum at c .
- 2) If $f'(a) < 0$ and $f'(b) > 0$, then there is a local minimum at c .
- 3) If $f'(a)$ and $f'(b)$, are both positive, or both negative, then there is no local extremum at c .

Problem. Find the intervals on which the function is increasing or decreasing $f(x) = 4x^3 - 9x^2 - 12x + 7$.

Problem. Use the first derivative test to determine the location of each local extremum of the function $f(x) = xe^{-x}$ and the value of the function at this extremum.

12.2 The Second Derivative & 12.4 Curve Sketching.

Definition. The second derivative of $y = f(x)$ can be written with any of the following notations:

$$f''(x), \quad y'', \quad \frac{d^2 y}{dx^2}, \quad \text{or} \quad D_x^2[f(x)]$$

The third derivative can be written in a similar way. For $n \geq 4$, the n th derivative is written $f^{(n)}(x)$.

Proposition. *Concavity Test.*

Let f be a function whose first and second derivatives exist at all points in the interval (a, b) .

- 1) If $f''(x) > 0$ for all x in (a, b) , then f is concave upward on (a, b) .
- 2) If $f''(x) < 0$ for all x in (a, b) , then f is concave downward on (a, b) .

If a function f has a point of inflection at $x = c$, then $f''(x) = 0$ or $f''(c)$ doesn't exist.

Problem. Find the critical numbers. Then use the second derivative test on each critical number to determine whether it leads to a local maximum or minimum, then find the inflection points.

$$f(x) = \frac{2}{3}x^3 + \frac{1}{2}x^2 - x - \frac{1}{4}$$

12.3 Optimization Applications.

Definition. Extreme Value Theorem.

If a function f is continuous on a closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on the interval. Each of these occurs either at an endpoint of the interval or at a critical number of f .

Problem. Find the absolute extrema for $f(x) = \frac{x-1}{x^2+1}$ on the interval $[1, 5]$.

Problem. A company wishes to manufacture a box with a volume of 36 cubic feet that is open on top and that is twice as long as it is wide. Find the dimensions of the box produced from the minimum amount of material.

Instructor: Jorge R. Viramontes
 MATHEMATICS 2301

13.1 Antiderivatives.

Definition. If $F'(x) = f(x)$, then $F(x)$ is an antiderivative of $f(x)$.

Proposition. If $F(x)$ and $G(x)$ are both antiderivatives of $f(x)$, then there is a constant C such that

$$F(x) - G(x) = C$$

Two antiderivatives of a function can differ only by a constant. E.g.

$$\begin{aligned}\frac{d(3x^2 + 5)}{dx} &= 6x \\ \frac{d(3x^2 + 2)}{dx} &= 6x\end{aligned}$$

where $F(x) = 3x^2 + 5$ & $G(x) = 3x^2 + 2$, so $F(x) - G(x) = (3x^2 + 5) - (3x^2 + 2) = 3$.

In this particular case the derivative is $f(x) = 6x$ and the antiderivative can be either $F(x) = 3x^2 + 5$ or $F(x) = 3x^2 + 2$, in general $F(x)$, represents the family of all antiderivatives of $f(x)$.

Definition. We define a new symbol for the derivative of $F(x)$, i.e. for $f(x)$, this will be $f(x) = \frac{dF(x)}{dx}$, where $dx \neq 0$, then

$$dF(x) = f(x) \cdot dx$$

Now, if we define another new symbol $\int f(x) dx$ as the family of all antiderivatives of the function f , we will have

$$\int dF(x) = \int f(x) \cdot dx = F(x)$$

or

$$\int f(x) \cdot dx = F(x)$$

The symbol \int is the **integral sign**, $f(x)$ is the **integrand**, and $\int f(x) \cdot dx$ is called an **indefinite integral**. For example, using this notation,

$$\int 6x dx = 3x^2 + C$$

Example. Suppose that $F(x) = 2x^5 + x^4 - 3x^2 + 8x + 1$, then $F'(x) = 10x^4 + 4x^3 - 6x + 8$, so we say that the derivative of $F(x)$ is $f(x) = 10x^4 + 4x^3 - 6x + 8$, or that $F(x) = 2x^5 + x^4 - 3x^2 + 8x + 1$ is the antiderivative of $f(x) = 10x^4 + 4x^3 - 6x + 8$, and we may write

$$\begin{aligned}\int f(x) dx &= F(x) \\ \int (10x^4 + 4x^3 - 6x + 8) dx &= 2x^5 + x^4 - 3x^2 + 8x + C\end{aligned}$$

Problem. Find each of the following.

1) $\int 25x \, dx$

2) $\int 5x^3 \, dx$

3) $\int 35 \, dx$

4) $\int (x^3 + 5x^2 - 10x - 4) \, dx$

5) $\int (10u^{3/2} - 14u^{5/2}) \, du$

6) $\int \left(\frac{3}{\sqrt{u}} + \frac{2u}{\sqrt{u}} \right) \, du$

7) $\int \left(\frac{9}{x} - 3e^x \right) \, dx$

Problem. The slope of the tangent line to a curve is given by

$$f'(x) = 6x^2 - 4x - 3$$

If the point $(0, 1)$ is on the curve, find the equation of the curve.

Instructor: Jorge R. Viramontes
MATHEMATICS 2301

13.2 Integration by Substitution.

Definition. If $u = f(x)$, the **differential** of u , written du , is defined as

$$du = f'(x) \cdot dx$$

E.g. If $u = 6x^4$, then $du = 24x^3 \cdot dx$. Some of the integrals contains integrands which are reminiscent of the chain rule and so we shall try to use differentials and the chain rule in reverse to find the antiderivative. This method of integration is called **integration by substitution**.

Problem. Find each of the following integrals, decide what factor should be u . Then find du .

1) $\int 2(4 + 2t)^3 dt.$

2) $\int \frac{4}{\sqrt{5u-1}} du.$

3) $\int \frac{3x^2 - 2}{(2x^3 - 4x)^{5/2}} dx.$

4) $\int 5ze^{-z^2} dz.$

5) $\int \frac{e^{1/z^2}}{z^3} dz.$

6) $\int \frac{e^{w+1}}{2 - e^{w+1}} dw.$

7) $\int \frac{x+2}{3x^2+12x+8} dx.$

8) $\int \frac{1}{x(\ln x)} dx.$

Instructor: Jorge R. Viramontes
 MATHEMATICS 2301

13.4 The Fundamental Theorem of Calculus.

Definition. The Definite Integral.

If f is a continuous function on the interval $[a, b]$, then the **definite integral** of f from a to b is the number

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} [(f(x_1) + f(x_2) + \cdots + f(x_n))\Delta x],$$

where $\Delta x = (b - a)/n$ and x_i is the left-hand endpoint of the i th interval.

Theorem. *The Fundamental Theorem of Calculus.*

Suppose f is continuous on the interval $[a, b]$ and F is any antiderivative of f .

Then

$$\int_a^b f(x)dx = F(b) - F(a)$$

Proposition. *Properties of Definite Integrals.*

- 1) $\int_a^b f(x)dx = 0$
- 2) $\int_a^b k \cdot f(x)dx = k \cdot \int_a^b f(x)dx$
- 3) $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$
- 4) $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx, \quad a \leq c \leq b$

Procedure. *Finding Area.*

To find the area bounded by $y = f(x)$, the vertical lines $x = a$ and $x = b$, and $x - axis$, use the following steps.

- 1.- Sketch the graph.
- 2.- Find any $x - intercept$ in $[a, b]$. These divide the total region into subregions.
- 3.- The definite integral will be positive for subregions above the $x - axis$ and negative for subregions below $x - axis$. Use separate integrals to find the areas of subregions.
- 4.- The total area is the sum of the areas of all subregions.

Problem. Evaluate the following definite integrals.

1. $\int_0^3 m^2(4m^3 - 2)^3 dm.$

2. $\int_1^2 \frac{\ln x}{x} dx.$

Problem. Find the area of each shaded region.

1.- .

2.-

13.1 Formulas.

Derivatives

Normal

$$\frac{dc}{dx} = 0$$

$$\frac{dx^n}{dx} = nx^{n-1}$$

$$\frac{d(u \pm v)}{dx} = \frac{du}{dx} \pm \frac{dv}{dx}$$

$$\frac{d(u \cdot v)}{dx} = u \cdot \frac{dv}{dx} + v \cdot \frac{du}{dx}$$

$$\frac{d(u/v)}{dx} = \frac{v \cdot \frac{du}{dx} - u \cdot \frac{dv}{dx}}{v^2}$$

$$\frac{de^x}{dx} = e^x$$

$$\frac{d \ln x}{dx} = \frac{1}{x}$$

Chain Rule

$$\frac{dc}{dx} = 0$$

$$\frac{du^n}{dx} = nu^{n-1} \cdot \frac{du}{dx}$$

$$\frac{de^u}{dx} = e^u \cdot \frac{du}{dx}$$

$$\frac{d \ln u}{dx} = \frac{1}{u} \cdot \frac{du}{dx}$$

Integrals

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \quad n \neq -1$$

$$\int [f(x) \pm g(x)] dx = \int f(x) dx \pm \int g(x) dx$$

$$\int k \cdot f(x) dx = k \cdot \int f(x) dx$$

$$\int \frac{dx}{x} = \ln x + c$$

$$\int a^x dx = \frac{a^x}{\ln a} + c$$

$$\int e^x dx = e^x + c$$

